Probability

The MEnTe Program

Math Enrichment through Technology

Title V East Los Angeles College

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Topics

Probability Introduction

When we speak of the probability of something happening, we are referring to the likelihood—or chances—of it happening. Do we have a better chance of it occurring or do we have a better chance of it not occurring?

Generally, we talk about this probability as a fraction, a decimal, or even a percent—

- the probability that if two dice are tossed the spots will total to seven is 1/6

- the probability that a baseball player will get a hit is .273

- the probability that it will rain is 20%

Empirical Probability

Some probabilities are determined from repeated experimentation and observation, recording results, and then using these results to predict expected probability. This kind of probability is referred to as empirical probability.

If we conduct an experiment and record the number of times a favorable event occurs, then the probability of the event occurring is given by:

$$P(E) = \frac{\text{\# of times event } E \text{ occurred}}{\text{total \# of times experiment performed}}$$

We can see this in the following example. If we flip a coin 500 times and it lands on heads 248 times, then the empirical probability is given by:

$$P(heads) = \frac{248}{500} \approx 0.5$$

Remember

$$P(E) = \frac{\#\text{ of times event } E \text{ occurred}}{\text{total }\#\text{ of times experiment performed}}$$

Theoretical Probability

Other probabilities are determined using mathematical computations based on possible results, or outcomes. This kind of probability is referred to as <u>theoretical probability</u>.

The theoretical probability of event *E* happening is given by:

$$P(E) = \frac{\text{\# of ways } E \text{ can occur}}{\text{total \# of possible outcomes}}$$

If we consider a fair coin has two sides and only one side is heads, and either side is likely to come up, then the theoretical probability of tossing heads is given by:

$$P(E) = \frac{\text{\# sides that are heads}}{\text{total number of sides}} = \frac{1}{2} = 0.5$$

Remember

$$P(E) = \frac{\text{\# of ways } E \text{ can occur}}{\text{total \# of possible outcomes}}$$

While in both cases illustrated for tossing a heads the probability comes out to be 0.5, it should be noted that empirical probability falls under the Law of Large Numbers which basically says that an experiment must be conducted a large number of times in order to determine the probability with any certainty.

You can flip a coin ten times and have heads come up seven times, but this does not mean that the probability is 0.7. The more times a coin is flipped, the more certainty we have to determine the probability of coming up heads.

Other examples of theoretical probability are found in determining the probability of drawing a certain card from a standard deck of cards.

A standard deck has four suits: spades (♠), hearts (♥), diamonds (♦), and clubs (♣). It has thirteen cards in each suit: ace, 2, 3, . . ., 10, jack, queen, and king. Each of these cards is equally likely to be drawn.

The probability of drawing a king is given by:

$$P(king) = \frac{\text{\# of kings in a deck}}{\text{total \# of cards in a deck}} = \frac{4}{52} = \frac{1}{13}$$

Remember

$$P(E) = \frac{\text{\# of ways } E \text{ can occur}}{\text{total \# of possible outcomes}}$$

The probability of drawing a heart is given by:

$$P(heart) = \frac{\text{\# of hearts in a deck}}{\text{total \# of cards in a deck}} = \frac{13}{52} = \frac{1}{4}$$

Remember

$$P(E) = \frac{\text{\# of ways } E \text{ can occur}}{\text{total \# of possible outcomes}}$$

The probability of drawing a face card (jack, queen, king) is given by:

$$P(face\ card) = \frac{\text{\# of face cards in a deck}}{\text{total \# of cards in a deck}} = \frac{12}{52} = \frac{3}{13}$$

Remember

$$P(E) = \frac{\text{\# of ways } E \text{ can occur}}{\text{total \# of possible outcomes}}$$

Dice (singular is die) are cubes that have spots on each side. The spots are usually numbered from 1 to 6. When a fair die is tossed, each side has an equally likely chance of ending up on top. The probability of tossing a die and having a 4 end up on top (this is called rolling a 4) is given by:

$$P(4) = \frac{\text{\# of faces with 4}}{\text{total \# of faces}} = \frac{1}{6}$$

The probability of tossing a die and rolling a 7 is given by:

$$P(7) = \frac{\text{\# faces with 7}}{\text{total \# of faces}} = \frac{0}{6} = 0$$

The probability of tossing a die and rolling a number less than 7 is given by:

$$P(\text{less than 7}) = \frac{\text{\# faces with less than 7}}{\text{total \# of faces}} = \frac{6}{6} = 1$$

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These examples lead to four rules or facts about probability:

1. The probability of an event that cannot occur is 0.
2. The probability of an event that must occur is 1.
3. Every probability is a number between 0 and 1 inclusive.
4. The sum of the probabilities of all possible outcomes of an experiment is 1.

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The complement of an event is all outcomes where the desired event does <u>not</u> occur. We can say the complement of E is *not* E (sometimes written as $\bar{E}$ or E').

Since any event will either occur or it will not occur, by rule 4 previously discussed, we get:

$$P(E) + P(not\ E) = 1$$

Remember
Rule 4: the sum of the probabilities of all possible outcomes of an experiment is 1.

$P(E) + P(not\ E) = 1$ can also be stated as:

$$P(not\ E) = 1 - P(E)$$

So the probability of tossing a die and not rolling a 4 is:

$$P(not\ 4) = 1 - P(4) = 1 - \frac{1}{6} = \frac{5}{6}$$

Compound Events

A compound event is an event consisting of two or more simple events. Examples of simple events are: tossing a die and rolling a 5, picking a seven from a deck of cards, or flipping a coin and having a heads show up.

An example of a compound event is tossing a die and rolling a 5 or an even number. The notation for this kind of compound event is given by $P(A \text{ or } B)$. This is the probability that event A or event B (or both) will occur.

In the case of rolling either a 5 or an even number on a die, the probability is arrived at by using the fact that there is only one way to roll a 5 and there are three ways to roll an even number.

So, out of the six numbers that can show up on top, we have four ways that we can roll either a 5 or an even number. The probability is given by:

$$P(5 \text{ or even}) = \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}$$

Probability of rolling a 5

Probability of rolling an even number

Notice however, if we want the probability of rolling a 5 or rolling a number greater than 3. There are three numbers greater than 3 on a die and one of them is the 5. We cannot count the 5 twice. The probability is given by:

$$P(5 \text{ or greater than } 3) = \frac{1}{6} + \left(\frac{3}{6} - \frac{1}{6}\right) = \frac{3}{6} = \frac{1}{2}$$

Probability of rolling a 5

Probability of rolling a number greater than 3

Probability of rolling the same 5

Addition Rule

This leads to the Addition Rule for compound events. The statement of this rule is that the probability of either of two events occurring is the probability for the first event occurring <u>plus</u> the probability for the second event occurring <u>minus</u> the probability of both event occurring simultaneously.

Stated mathematically the rule is given by:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Thus, the probability of drawing a 3 or a club from a standard deck of cards is:

$$P(3 \text{ or club}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Cards with a 3

Cards with clubs

Card that is a 3 and a club

If two events are mutually exclusive, they cannot occur simultaneously. Therefore, $P(A \text{ or } B) = 0$, and the Addition Rule for mutually exclusive events is given by:

$$P(A \text{ or } B) = P(A) + P(B)$$

Multiplication Rule for Independent Events

Independent events are events in which the occurrence of the events will not affect the probability of the occurrence of any of the other events.

When we conduct two independent events we can determine the probability of a given outcome in the first event followed by another given outcome in the second event.

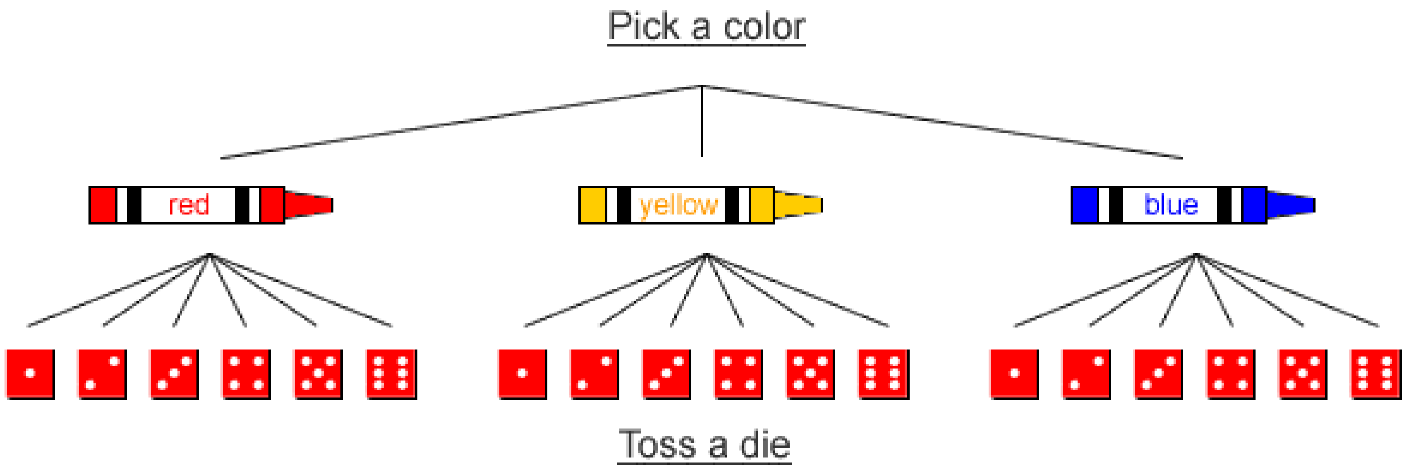
An example of this is picking a color from a set of crayons, then tossing a die. Separately, each of these events is a simple event and the selection of a color does not affect the tossing of a die.

If the set of crayons consists only of red, yellow, and blue, the probability of picking red is $\frac{1}{3}$. The probability of tossing a die and rolling a 5 is $\frac{1}{6}$. But the probability of picking red and rolling a 5 is given by:

$$P(\text{red and } 5) = P(\text{red}) \cdot P(5)$$

$$= \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$$

This can be illustrated using a “tree” diagram.



Since there are three choices for the color and six choices for the die, there are eighteen different results. Out of these, only one gives a combination of red and 5. Therefore, the probability of picking a red crayon and rolling a 5 is given by:

$$P(\text{red and } 5) = P(\text{red}) \cdot P(5)$$

$$= \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$$

The multiplication rule for independent events can be stated as:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

This rule can be extended for more than two independent events:

$$P(A \text{ and } B \text{ and } C, etc.) = P(A) \cdot P(B) \cdot P(C), etc.$$

Multiplication Rule for Dependent Events

Dependent events are events that are not independent. The occurrence of one event affects the probability of the occurrence of other events. An example of dependent events is picking a card from a standard deck then picking another card from the remaining cards in the deck.

For instance, what is the probability of picking two kings from a standard deck of cards? The probability of the first card being a king is $\frac{4}{52} = \frac{1}{13}$. However, the probability of the second card depends on whether or not the the first card was a king.

If the first card was a king then the probability of the second card being a king is $\frac{3}{51} = \frac{1}{17}$.

If the first card was not a king, the probability of the second card being a king is $\frac{4}{51}$.

Therefore, the selection of the first card affects the probability of the second card.

When we are looking at probability for two dependent events we need to have notation to express the probability for an event to occur given that another event has already occurred.

If A and B are the two events, we can express the probability that B will occur if A has already occurred by using the notation:

$$P(B|A)$$

This notation is generally read as "the probability of B, given A.

The multiplication rule can now be expanded to include dependent events. The rule now reads:

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Of course, if *A* and *B* are independent, then:

$$P(B|A) = P(B)$$

As an example, in a group of 25 people 16 of them are married and 9 are single. What is the probability that if two people are randomly selected from the group, they are both married?

If A represents the first person chosen is married and B represents the second person chosen is married then:

$$P(A \text{ and } B) = \frac{16}{25} \cdot \frac{15}{24} = \frac{2}{5}$$

Here, $P(B|A)$ is now the event of picking another married person from the remaining 15 married persons. The probability for the selection made in B is affected by the selection in A.

Counting Principles

Sometimes determining probability depends on being able to count the number of possible events that can occur, for instance, suppose that a person at a dinner can choose from two different salads, five entrees, three drinks, and three desserts. How many different choices does this person have for choosing a complete dinner?

The Multiplication Principal for counting (which is similar to the Multiplication Principle for Probability) says that if an event consists of a sequence of choices, then the total number of choices is equal to the product of the numbers for each individual choice.

If $c_1, c_2, c_3, \ldots, c_n$, represent the number of choices that can be made for each option then the total number of choices is:

$$c_1 \cdot c_2 \cdot c_3 \ \ldots \cdot c_n$$

For our person at the dinner, the total number of choices would then be $2 \cdot 5 \cdot 3 \cdot 3 = 90$ different choices for combining salad, entrée, drink, and dessert.

Odds

Odds are related to probability, but there are slightly different computing rules for figuring out odds. The odds of an event occurring is given by:

$$\text{Odds in favor of an event} = \frac{P(E)}{P(not\ E)}$$

And the Odds of an event <u>not</u> occurring is given by:

$$\text{Odds against an event} = \frac{P(not\ E)}{P(E)}$$

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Notice that these are reciprocals of each other and the odds for an event not happening are not determined by subtracting from 1, as in the case for determining the probability of an event not happening.

Odds against an event $= \dfrac{P(not\ E)}{P(E)}$

Probability of an event not happening $P(not\ E) = 1 - P(E)$

Permutations

A permutation is an arrangement of objects where order is important. For instance the digits 1,2, and 3 can be arranged in six different orders --- 123, 132, 213, 231, 312, and 321. Hence, there are six permutations of the three digits. In fact there are six permutations of any three objects when all three objects are used.

In general the number of permutations can be derived from the Multiplication Principal. For three objects, there are three choices for selecting the first object. Then there are two choices for selecting the second object, and finally there is only one choice for the final object. This gives the number of permutations for three objects as $3 \cdot 2 \cdot 1 = 6$.

Now suppose that we have 10 objects and wish to make arrangements by selecting only 3 of those objects. For the first object we have 10 choices. For the second we have 9 choices, and for the third we have 8 choices. So the number of permutations when using 3 objects out of a group of 10 objects is $10 \cdot 9 \cdot 8 = 720$.

We can use this example to help derive the formula for computing the number of permutations of r objects chosen from n distinct objects $r \leq n$. The notation for these permutations is $P(n,r)$ and the formula is:

$$P(n,r) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot [n-(r-1)]$$

We often use factorial notation to rewrite this formula. Recall that:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

And $0! = 1$

Using this notation we can rewrite the Permutation Formula for $P(n,r)$ as

$$P(n,r) = \frac{n!}{(n-r)!}$$

It is important to remember that in using this formula to determine the number of permutations:

1. The *n* objects must be distinct
2. That once an object is used it cannot be repeated
3. That the order of objects is important.

Combinations

A combination is an arrangement of objects in which order is not important. We arrange r objects from among n distinct objects where $r \leq n$. We use the notation C(n, r) to represent this combination. The formula for C(n, r) is given by:

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

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The Combination Formula is derived from the Permutation Formula in that for a permutation every different order of the objects is counted even when the same objects are involved. This means that for r objects, there will be $r!$ different order arrangements.

So in order to get the number of different combinations, we must divide the number of permutations by r!. The result is the value we get for C(*n*, *r*) in the previous formula.

Permutation	Combination
$P(n,r)=\dfrac{n!}{(n-r)!}$	$C(n,r)=\dfrac{n!}{(n-r)!\,r!}$

Permutations of Repeated Objects

It is possible that in a group of objects some of the objects may be the same. In taking the permutation of this group of objects, different orders of the objects that are the same will not be different from one another.

In other words if we look at the group of letters in the word ADD and use D_1 to represent the first D, and D_2 to represent the second, we can then write the different permutations as AD_1D_2, AD_2D_1, D_1AD_2, D_2AD_1, D_1D_2A, and D_2D_1A.

But if we substitute the *D*s back for the D_1 and D_2, then AD_1D_2 and AD_2D_1 both appear as *ADD*, and the six permutations become only three *distinct* permutations.

Therefore we will need to divide the number of permutations by 2 to get the number of distinct permutations.

In permutations of larger groups of objects, the division becomes a little more complicated.

To explain the process, let us look at the word WALLAWALLA. This word has 4 A's, 4 L's, and 2 W's.

Consider that there are 10 locations for each of these letters. These 10 locations will be filled with 4 A's, and since the A's are all the same, the order in which we place the A's will not matter. So if we are filling 10 locations with 4A's the number of ways we can do this is $C(10, 4)$.

Remember
$$C(n,r) = \frac{n!}{(n-r)!r!}$$

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Once these 4 locations have been filled, there remain 6 locations to fill with the 4 L's. These can be filled in $C(6,4)$ ways, and the last 2 locations are filled with the W's in $C(2,2)$ ways.

Finally, we multiply these together to get

$$C(10,4)\cdot C(6,4)\cdot C(2,2)=\frac{10!}{4!\cdot 6!}\cdot\frac{6!}{4!\cdot 2!}\cdot\frac{2!}{2!\cdot 0!}$$

$$=\frac{10!}{4!\cdot 4!\cdot 2!\cdot 1}$$

$$=\frac{10!}{4!\cdot 4!\cdot 2!}$$

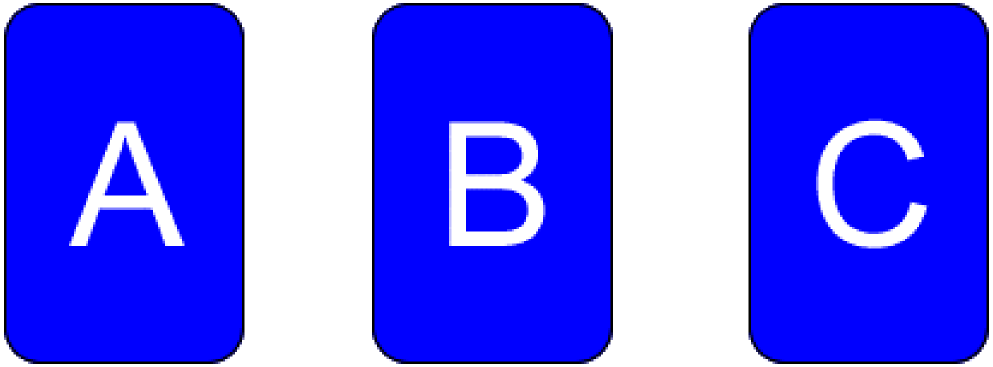
This leads to the general formula for permutations involving n objects with n_1 of one kind, n_2 of a second kind, …and n_k of a kth kind.

The number of permutations in this case is:

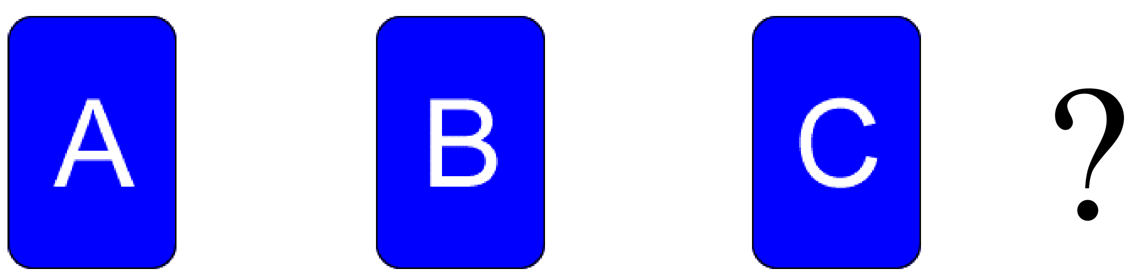
$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

where $n=n_1+n_2+\ldots n_k$.

Counting other choices sometimes requires a bit more reasoning to determine how many possibilities there are.



Suppose there are three cards that are each marked with a different letter, A, B, or C. If the cards are face down, and a person can pick one, two or all three of the cards, what is the possibility that the person will pick up the card with the letter A on it?



In this case there are three ways that one card can be picked. Out of these there is only one possibility of picking the *A*.

C First way

B Second way

A Third way *A* is picked!

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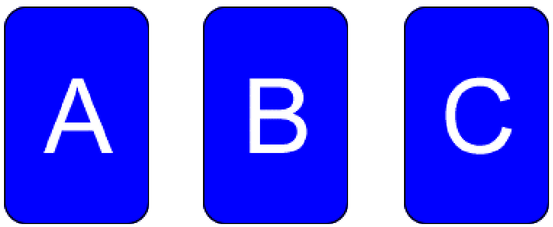
There are <u>three ways</u> of picking two cards. Out of these three pairs, there are two that will include the *A*.

B C — First way

A B — Second way *A* is picked!

A C — Third way *A* is picked!

There is only <u>one way</u> to pick all three cards, and of course, if all three cards are picked, the *A* will always be included.



A is picked!

So there are a total of seven ways the cards can be picked if the person can pick one, two, or all three cards. Of these choices, four of them will include the *A*, so the probability that the *A* will be picked is:

$$\frac{4}{7}$$

4 — Possibilities of picking the A card

7 — Total # of ways to pick the three cards

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